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## SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Let  $x^2, y^2, z^2$  be three square numbers in arithmetical progression; then we must have

$$y^2 - x^2 = z^2 - y^2, \quad \text{or} \quad x^2 + z^2 = 2y^2. \quad (1)$$

Assume  $z = v + w$ ,  $x = v - w$ , and (1) becomes after dividing by 2,

$$v^2 + w^2 = y^2. \quad (2)$$

Take now  $v = p^2 - q^2$ ,  $w = 2pq$ , and (2) is satisfied. Retracing, we find

$$z = p^2 - q^2 + 2pq, \quad x = p^2 - q^2 - 2pq, \quad y = p^2 + q^2.$$

Hence, the required squares are

$$x^2 = (p^2 - q^2 - 2pq)^2, \quad y^2 = (p^2 + q^2)^2, \quad z^2 = (p^2 - q^2 + 2pq)^2.$$

Taking  $p = 2$ ,  $q = 1$ , the numbers are 1, 25, 49; taking  $p = 3$ ,  $q = 2$ , the numbers are 49, 169, 289; taking  $p = 4$ ,  $q = 1$ , the numbers are 49, 289, 529; taking  $p = 4$ ,  $q = 3$ , the numbers are 289, 625, 961; and so on, indefinitely.

The common difference of three square numbers in arithmetical progression can not be a square number. See Barlow's "Theory of Numbers," p. 257. An equivalent theorem is also given in Carmichael's *Diophantine Analysis*, p. 14.

There can not be *four* square numbers in arithmetical progression. Barlow, same page. Therefore there can not be *five*, nor any greater number than three, of squares in arithmetical progression.

Also solved by J. L. RILEY, V. M. SPUNAR, and the PROPOSER.

**265 (Number Theory). Proposed by J. W. NICHOLSON, Louisiana State University.**

If the roots of  $x^4 - ax^2 + bx + c = 0$  are rational, prove that  $4(a + yz) - 3(y + z)^2$  is a perfect square,  $y$  and  $z$  being any two roots of the equation.

## SOLUTION BY N. P. PANDYA, Sojitra, India.

Since  $y$  and  $z$  are roots of the given equation,  $x^2 - x(y + z) + yz$  is a factor of the left-hand side of the equation.

Since the term in  $x^3$  is wanting, the remaining roots are given by

$$x^2 + x(y + z) + \frac{c}{yz} = 0. \quad (1)$$

The product of (1) with  $x^2 - x(y + z) + yz = 0$  gives

$$a = -\frac{c}{yz} - yz + (y + z)^2.$$

Hence,

$$4(a + yz) = -\frac{4c}{yz} + 4(y + z)^2,$$

or

$$4(a + yz) - 3(y + z)^2 = (y + z)^2 - \frac{4c}{yz} = \text{a square},$$

since the roots of (1) are rational and its discriminant is therefore a square.

**267 (Number Theory). Proposed by C. C. YEN, Tangshan, North China.**

A number theory function  $\phi(n)$  is defined for every positive integer  $n$ , and for every such number  $n$  it satisfies the relation  $\phi(d_1) + \phi(d_2) + \phi(d_3) + \cdots + \phi(d_r) = n$ , where  $d_1, d_2, \dots, d_r$  are the divisors of  $n$ . From this property alone show that

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right),$$

where  $p_1, p_2, \dots, p_k$  are the different prime factors of  $n$ .

SOLUTION BY OLIVE C. HAZLETT, Bryn Mawr College.

The theorem is clearly true for all primes. Accordingly, assume the theorem is true for all divisors of  $n = \prod_{i=1}^k p_i^{e_i}$  which are less than  $n$ . Now for  $n$  the defining equation becomes

$$\phi(n) + A_n + p_1^{e_1} \left(1 - \frac{1}{p_1}\right) B_n = n,$$

where  $A_n$  is the sum of the  $\phi$ -functions formed for the divisors of  $p_1^{e_1-1} p_2^{e_2} \cdots p_k^{e_k}$  and  $B_n$  is a similar sum formed for all distinct factors of any of the numbers  $p_2^{e_2-1} p_3^{e_3} \cdots p_k^{e_k}$ ,  $p_2^{e_2} p_3^{e_3-1} p_4^{e_4} \cdots p_k^{e_k}$ ,  $\dots$ . It is easy to find an expression for  $A_n$ , but it is sufficient for our purposes to note that  $A_n$  is a polynomial in  $p_2, \dots, p_k$  of degree at most  $\sum_{i=2}^k e_i - 1$ . Therefore  $p_1^{e_1} \left(1 - \frac{1}{p_1}\right)$  is a factor of  $\phi(n)$ . Since this proof is perfectly general, it holds for every expression of the form  $p_i^{e_i} \left(1 - \frac{1}{p_i}\right)$  ( $i = 1, \dots, p$ ), and thus  $\prod_{i=1}^k p_i^{e_i} \left(1 - \frac{1}{p_i}\right)$  is a factor of  $\phi(n)$ . Comparing the coefficients of  $\prod_{i=1}^k p_i^{e_i}$  our formula is proved.

Also solved by H. C. FEEMSTER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO FINDING DERIVATIVES OF TRIGONOMETRICAL FUNCTIONS.

By T. H. HILDEBRANDT, University of Michigan.

In most textbooks on the elementary calculus the derivatives of the trigonometric functions are based on the derivative of the sine function, which, in turn, is derived from the definition of derivative. The proofs dealing with the value of this derivative seem to have something indirect about them. All goes well until the point is reached where the expression

$$\lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}$$

is to be evaluated, and then one of two methods is used. Either  $\sin(x + \Delta x)$  is expanded by the formula for the sine of the sum of two angles and the formula for  $1 - \cos x$  in terms of half angles is used, or the formula for the difference of two sines is used. Both of these latter formulæ have long since escaped the memory of the average sophomore student—if they ever had lodging there—and he practically accepts this part of the derivation on faith.

While it must be admitted that the most natural beginning for a chapter on the derivatives of trigonometrical functions is a paragraph devoted to finding the derivative of the sine, this advantage is more than counterbalanced by the simplicity with which it is possible to obtain the derivative of the tangent function directly from the definition of derivative—a fact which seems almost to have